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Numerical Methods - MA 207
Finite Differences

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1. Evaluate $\Delta \tan^{-1} x, \Delta(e^x \log 2x), \Delta(x^2/\cos 2x)$.
2. Evaluate
$$\Delta\left[\frac{5x+12}{x^2+5x+16}\right],$$
interval of differencing being unity.
3. Find the successive differences of $f(x) = ab^{cx}$ and sum the first n differences.
4. Find the function (with suitable h) whose first difference is
   (a) $ax + b$
   (b) $\sin x$
   (c) e^x
   (d) e^{a+bx}
5. Prove that the operator Δ obeys the **index law** :
$$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$$
where m and n are positive integers.
6. Evaluate $E^0 f(x), \Delta(c)$ and $E(c)$ where c is a constant.
7. True or false: If $\Delta f(x) = 0$, then either $\Delta \equiv 0$ or $f(x) = 0$.
8. True or false: $E^2 f(x)$ and $E^2[f(x)]^2$ are identical.
9. An operator T is said to be **linear** if
$$T[af(x) + bg(x)] = aT[f(x)] + bT[g(x)],$$
where a, b are constants. Prove that the operators E and Δ are linear.
10. Prove that for all integral values of n, $f(a + nh) = \sum_{r=0}^{n} \binom{n}{r} \Delta^r f(a)$ with the help of the operators E and Δ.
11. If $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \quad a_0 \neq 0$, then find $\Delta^n f(x)$. Obtain $\Delta^{25}\{(x-a)(x-b)\cdots(x-z)\}$ where the operand has only 25 factors and there is no factor of the type $(x - x)$.
12. Prove that
$$e^x = \left(\frac{\Delta^2}{E}\right) e^x . \frac{Ee^x}{\Delta^2 e^x},$$
the interval of differencing being unity.
13. Prove that $y_n - y_0$ is the sum of all entries in the first difference column of the difference table for (x_i, y_i), $0 \leq i \leq n$.
$$\text{That is,} \quad \sum_{x=0}^{n-1} \Delta y_x = y_n - y_0.$$
14. Prove that $y_4 = y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1$.

15. Show that $y_4 = y_0 + 4\Delta y_0 + 6\Delta^2 y_{-1} + 10\Delta^3 y_{-1}$ if fourth and higher differences are zero.

16. Prove that
$$y_0 + y_1 + \cdots + y_n = \binom{n+1}{1} y_0 + \binom{n+1}{2} \Delta y_0 + \cdots + \binom{n+1}{n+1} \Delta^n y_0.$$

17. Prove the following identity
$$\sum_{x=0}^{\infty} y_{2x} = \frac{1}{2} \sum_{x=0}^{\infty} y_x + \frac{1}{4}\left(1 - \frac{\Delta}{2} + \frac{\Delta^4}{4} - \cdots\right) y_0.$$

18. Find the sum of the series,
$$1.2\ \Delta x^n - 2.3\ \Delta^2 x^n + 3.4\ \Delta^3 x^n - 4.5\ \Delta^4 x^n + \cdots\cdots$$
the interval of differencing being unity.

19. Deduce the following:

(a) $\nabla \equiv 1 - E^{-1}$

(b) $\Delta - \nabla \equiv \Delta\nabla = \delta^2$

(c) $\Delta \equiv E^{1/2} - E^{-1/2}$

(d) $\mu \equiv (1/2)(E^{1/2} + E^{-1/2})$

(e) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} \equiv 2 + \Delta$

(f) $\Delta \equiv \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$

(g) $1 + \delta^2\mu^2 = \left(1 + \frac{\delta^2}{2}\right)$

(h) $\mu^2 \equiv 1 + (1/4)\Delta^2$

(i) $E\nabla \equiv \nabla E \equiv \Delta \equiv \Delta E^{1/2}$.

(j) $\mu = \frac{2+\Delta}{2\sqrt{1+\Delta}} + \sqrt{1 + \frac{\delta^2}{4}}$.

20. If $y = a(3)^x + b(-2)^x$ and $h = 1$, prove that $(\Delta^2 + \Delta - 6)y = 0$.

21. Evaluate $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$.

22. Show that $\Delta\left[\frac{1}{f(x)}\right] = \frac{\Delta f(x)}{f(x)f(x+1)}$.

23. Prove that $\Delta \log f(x) = \log\left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$.

24. Find the missing y_x values from the first differences provided.

y_x	0	-	-	-	-	-
Δy_x	0	1	2	4	7	11

25. Prove the following identities.

(a) $E \equiv e^{hD}$ (Hint: Taylor's series)

(b) $hD \equiv \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots$

(c) $hD \equiv \log(1+\Delta) \equiv -\log(1-\nabla) \equiv \sinh^{-1}(\mu\delta)$

(d) $\nabla y_{n+1} \equiv h(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \cdots)y'_n$.

26. Using the method of separation of symbols, prove the following identities:

(a) $y_x - \Delta^n y_{x-n} = y_{x-1} + \Delta y_{x-2} + \Delta^2 y_{x-3} + \cdots + \Delta^{n-1} y_{x-n}$.

(b) $y_1\, x + y_2\, x^2 + y_3\, x^3 + \cdots = \frac{x}{x-1} y_1 + \frac{x^2}{(1-x)^2}\Delta y_1 + \frac{x^3}{(1-x)^3}\Delta^2 y_1 + \cdots$.

27. Taking fifth order differences of y_x to be constant and given $y_0, y_1, y_2, y_3, y_4, y_5$ prove that
$$y_{2\frac{1}{2}} = \frac{1}{2}c + \frac{25(c-b) + 3(a-c)}{256}$$
where $a = y_0 + y_5,\ b = y_1 + y_4,\ c = y_2 + y_5$.

28. For any positive integer n, prove the following:

 (a) $\Delta^r \binom{n}{n} = \binom{x}{n-r}, \quad r < n$
 (b) $\Delta^n \binom{x}{n} = 1.$

29. Obtain the function whose first difference is $2x^3 - 3x^2 + 3x - 10$.

30. If $y = \frac{1}{(3x+1)(3x+4)(3x+7)}$, evaluate $\Delta^2 y$. Also find $\Delta^{-1} y$.

31. If $y_0 = 3, y_{11} = 6, y_{12} = 11, y_{13} = 18, y_{14} = 27$, find y_4.

32. If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -784, y_2 + y_6 = 686, y_3 + y_5 = 1088$, find y_4.

33. Using the method of separation of symbols, prove that $y_0 + \frac{y_1}{1!}x + \frac{y_2}{2!}x^2 + \frac{y_3}{3!}x^3 + \cdots = e^x[y_0 + x\Delta y_0 + \frac{x^2}{2!}\Delta^2 y_0 + \frac{x^3}{3!}\Delta^3 y_0 + \cdots]$. Hence find the sum of the following series.

 (a) $1^3 + \frac{2^3}{1!}x + \frac{3^3}{2!}x^2 + \frac{4^3}{3!}x^3 + \cdots \infty$
 (b) $1 + \frac{4x}{1!} + \frac{10x^2}{2!} + \frac{20x^3}{3!} + \frac{35x^4}{4!} + \frac{56x^5}{5!} \cdots \infty.$

34. Prove that for any positive integer n,

$$[x]^n = (x - (n-1)h)[x]^{n-1}.$$

35. Express $2x^3 - 3x^2 + 3x - 10$ in factorial notation by both the methods.

36. **Fill the blank:** The coefficient of the highest power of x ________________(remains unchanged / may change) while transforming a polynomial to factorial notation.

37. If $f(x) = (2x+1)(2x+3)\cdots(2x+15)$, find the value of $\Delta^4 f(x)$.

38. Express the function

$$x^4 - 12x^3 + 24x^2 - 30x + 9$$

 in factorial notation, the interval of differencing being unity.

39. Using factorial notation, obtain the function whose first difference is

$$x^3 + 4x^2 + 9x + 12.$$

40. Express $2x^3 - 3x^2 + 3x - 10$ and its successive difference in factorial notation.

41. A second degree polynomial passes through $(0,1), (1,3), (2,7)$ and $(3,13)$. Find the polynomial.

42. Prove that $[x]^r[x - rh]^n = [x]^{r+n}$.

43. Find the relation between α, β and γ in order that $\alpha + \beta x + \gamma x^2$ may be expressible in one term in the factorial notation.

44. One entry in the following table is incorrect and y is a cubic polynomial in x. Use the difference table to locate and correct the error.

x	0	1	2	3	4	5	6	7
y	25	21	18	18	27	45	76	123

45. The following table gives the values of y which is a polynomial of degree five. It is known that $f(x)$ is in error.

x	0	1	2	3	4	5	6	7
y	25	21	18	18	27	45	76	123

46. Find the missing term in the table:

x	2	3	4	5	6
y	45	49.2	54.1	-	-67.4

47. Find the missing terms in the following data.

x	45	50	55	60	65
y	3	-	2	-	-2.4

48. Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values.

x	0	1	2	3	4	5	6	7
y	1	-1	1	-1	-	-	-	-

49. Find the sum to n terms of the series.

   (a) $2.3.4 + 3.4.5 + 4.5.6 + \cdots$

   (b) $\frac{1}{3.4.5} + \frac{1}{4.5.6} + \frac{1}{5.6.7} + \cdots$

   (c) $1^3 + 2^3 + \cdots + n^3$.

50. Prove **Montmort's theorem** that

$$y_0 + y_1 x + y_2 x^2 + \cdots \infty = \frac{y_0}{1-x} + \frac{x \Delta y_0}{(1-x)^2} + \frac{x^2 \Delta^2 y_0}{(1-x)^3} + \cdots \infty.$$

Hence find the sum of the series

$$1.2 + 2.3x + 3.4x^2 + \cdots \infty.$$

51. Using Montmort's theorem find the sum of the series

   (a) $1.3 + 3.5x + 5.7x^2 + 7.9x^3 + \cdots \infty$.

   (b) $1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots \infty$.

52. Match the following :

$E\nabla$	$\frac{\Delta+\nabla}{2}$
hD	$\Delta - \nabla$
$\nabla\Delta$	Δ
$\mu\delta$	$-\log(1-\nabla)$

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